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Note

A Note on the Zero Distribution of Orthogonal Polynomials

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A result is proved which, when combined with a lemma of P. Nevai, leads to a generalization of some theorems about the asymptotic distribution of the zeros of generalized orthogonal polynomials © 1991 Academic Press, Inc.

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Let $p_n(x)=\gamma_n x^n+\cdots(\gamma_n>0)$ $(n=0, 1, 2, ...)$ be a sequence of polynomials satisfying

$$\begin{gathered} xp_n(x)=\frac{\gamma_n}{\gamma_{n+1}}p_{n+1}(x)+\alpha_n p_n(x)+\frac{\gamma_{n-1}}{\gamma_n}p_{n-1}(x) \\ p_{-1}=\gamma_{-1}=0, p_0(x)=\gamma_0, \alpha_n\in\mathbb{R}, \gamma_n>0 \qquad (n=0, 1, 2, ...). \end{gathered} \tag{1.1}$$

According to a theorem of Favard [1] there will be a distribution function $\alpha(x)$ such that

$$\int_{-\infty}^{+\infty} p_m(x)\, p_n(x)\, d\alpha(x)=\delta_{m,n}.$$

Next let $q_n(x) = \delta_n x^n + \cdots (\delta_n > 0)$ $(n = 0, 1, 2, ...,)$ be a second sequence of polynomials satisfying a recurrence relation like (1.1) and a similar orthogonality relation but with α_n, γ_n, $\alpha(x)$ replaced by β_n, δ_n, $\beta(x)$, respectively The distribution function $\alpha(x)$, for example, is substan-

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tially unique [2, p. 58] if the sequences $\{\alpha_n\}_0^\infty$ and $\{\gamma_n/\gamma_{n+1}\}_0^\infty$ are bounded and this is the case if and only if the support of $d\alpha$,

$$\operatorname{supp}(d\alpha)=\{x : \alpha(x-\varepsilon)<\alpha(x+\varepsilon)\ \forall \varepsilon>0\}$$

is compact.

Throughout the rest of this note it is supposed that both $\operatorname{supp}(d\alpha)$ and $\operatorname{supp}(d\beta)$ are compact. The smallest intervals containing these are Δ_1 and Δ_2, respectively, and we write $\Gamma=\Delta_1\cup\Delta_2$. The zeros of p_n and q_n (all of which are simple and lie in the corresponding Δ_ν) are denoted by $x_{kn}(d\alpha)$ and $x_{kn}(d\beta)$ $(1\leqslant k\leqslant n : n=1, 2, \ldots)$.

DEFINITION. Let a_{kn}, b_{kn} $(1\leqslant k\leqslant n : n=1, 2, \ldots)$ be two triangular arrays of numbers, all contained in a compact interval Δ of the real axis. If, for each $f\in C(\Delta)$ we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\,[f(a_{kn})-f(b_{kn})]=0$$

then we say that these arrays are "equally distributed."

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The object of this note is to prove the following result, which has some interesting consequences.

THEOREM 1. *Let $p_n(x)$ and $q_n(x)$ be as defined in Section* 1 *with* $\operatorname{supp}(d\alpha)$ *and* $\operatorname{supp}(d\beta)$ *both compact. Then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\left\{|\alpha_k-\beta_k|+\left|\frac{\gamma_{k-1}}{\gamma_k}-\frac{\delta_{k-1}}{\delta_k}\right|\right\}=0 \tag{2.1}$$

if and only if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\left|\int_{\Delta_1} fp_kp_{k+l}\,d\alpha-\int_{\Delta_2} fq_kq_{k+l}\,d\beta\right|=0 \tag{2.2}$$

for each fixed integer $l\geqslant 0$ and each $f\in C(\Gamma)$.

The following lemma was proved by P. Nevai in [3, Lemma 5.1].

LEMMA A. *If* $\operatorname{supp}(d\alpha)$ *is compact and $f\in C(\Delta_1)$ then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left\{f(x_{kn}(d\alpha))-\int_{\Delta_1} fp_{k-1}^2\,d\alpha\right\}=0.$$

Combining these two results we obtain the following:

THEOREM 2. *Let* $p_n(x)$ *and* $q_n(x)$ *be as defined in Section* 1 *with* $\operatorname{supp}(d\alpha)$ *and* $\operatorname{supp}(d\beta)$ *both compact. Let*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\left\{|\alpha_k-\beta_k|+\left|\frac{\gamma_{k-1}}{\gamma_k}-\frac{\delta_{k-1}}{\delta_k}\right|\right\}=0.$$

Then the zeros $x_{kn}(d\alpha)$ *and* $x_{kn}(d\beta)$ $(1\leqslant k\leqslant n: n=1,2,...)$ *are equally distributed.*

To illustrate this theorem, let us take $\beta_n=a (n\geqslant 0)$, $\delta_0=1/\sqrt{\pi}$, $\delta_n=(1/\sqrt{2\pi})(2/b)^n$ $(n\geqslant 1)$ so that the q polynomials are the orthonormal, first kind, Chebychev polynomials for the interval $[a-b, a+b]$. Then Theorem 2 gives the following result.

THEOREM 3. *If* $\operatorname{supp}(d\alpha)$ *is compact and* $f\in C(\Delta_1\cup[a-b, a+b])$ *and if*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\left\{|\alpha_k-a|+\left|\frac{\gamma_{k-1}}{\gamma_k}-\frac{b}{2}\right|\right\}=0 \qquad (b>0)$$

then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(x_{kn}(d\alpha))=\frac{1}{\pi}\int_{a-b}^{a+b}\frac{f(t)\,dt}{\sqrt{b^2-(t-a)^2}}.$$

This theorem follows easily from Theorem 2 since we see that with the above choice of $q_n(x)$ we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\int_{\Delta_2} fq_{k-1}^2\,d\beta$$

$$=\lim_{n\to\infty}\int_{\Delta_2} fq_{n-1}^2\,d\beta=\frac{1}{\pi}\int_{a-b}^{a+b}\frac{f(t)\,dt}{\sqrt{b^2-(t-a)^2}}.$$

Then Lemma A, applied to the q polynomials, and Theorem 2 give the result.

Theorem 3 generalizes a theorem in [3, Theorem 5.3] which had the hypotheses

$$\alpha_n\to a, \qquad \frac{\gamma_{n-1}}{\gamma_n}\to\frac{b}{2} \qquad (b>0)$$

In a similar way, suppose that we define

$$\beta_n=\begin{cases}c_1 (n \text{ even})\\ c_2 (n \text{ odd}),\end{cases} \qquad \frac{\delta_{n-1}}{\delta_n}=\begin{cases}d_1\geqslant 0\ (n \text{ even})\\ d_2\geqslant 0\ (n \text{ odd})\end{cases} \qquad (n\geqslant 1)$$

and suppose that $\{\alpha_n\}_0^\infty$ and $\{\gamma_{n-1}/\gamma_n\}_0^\infty$ satisfy (2.1). Then Theorem 2 yields a generalization of a result due to W. Van Assche [4, Theorem 4] in which it was assumed that

$$\alpha_n \to \begin{cases} c_1(n \text{ even}) \\ c_2(n \text{ odd}), \end{cases} \qquad \frac{\gamma_{n-1}}{\gamma_n} \to \begin{cases} d_1 \geqslant 0 & (n \text{ even}) \\ d_2 \geqslant 0 & (n \text{ odd}) \end{cases}$$

instead of the weaker (2.1). Similar remarks apply to the cases in which $\{\beta_n\}_0^\infty$ ad $\{\delta_{n-1}/\delta_n\}_0^\infty$ would be periodic with period $N > 2$. Of course, in those cases, the explicit evaluation of the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_{kn}(d\beta))$$

in the form of an integral would be more complicated. In this connection we refer to [5].

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We now present the proof of Theorem 1.

Proof (a). Equation (2.2) implies (2.1).

Take $f(x) = x$ and $l = 0$ in (2.2) and then take $f(x) = x$ and $l = 1$ there. We obtain (2.1) by virtue of the recurrence relations satisfied by p_n and q_n

Proof (b). Equation (2.1) implies (2.2).

It is enough to prove this for a polynomial defined on Γ and so enough to take $f(x) = x^m$ $(m = 0, 1, 2, ...)$. We proceed by induction on m. By the orthonormality of the two sequences of polynomials p_n and q_n the result is true for $m = 0$ and any fixed integer $l \geqslant 0$. Assume, then, that (2.2) is true for $f(x) = x^M$ for some $M \geqslant 0$ and all fixed integers $l \geqslant 0$. From the recurrence relations satisfied by the p_n and q_n we obtain

$$\begin{aligned}
&\frac{1}{n}\sum_{k=0}^{n-1}\left|\int_{\Delta_1} x^{M+1} p_k p_{k+l}\,d\alpha - \int_{\Delta_2} x^{M+1} q_k q_{k+l}\,d\beta\right| \\
&\quad\leqslant \frac{1}{n}\sum_{k=0}^{n-1}\left|\frac{\gamma_k}{\gamma_{k+1}}\int_{\Delta_1} x^M p_{k+1} p_{k+l}\,d\alpha - \frac{\delta_k}{\delta_{k+1}}\int_{\Delta_2} x^M q_{k+1} q_{k+l}\,d\beta\right| \\
&\quad\quad + \frac{1}{n}\sum_{k=0}^{n-1}\left|\alpha_k\int_{\Delta_1} x^M p_k p_{k+l}\,d\alpha - \beta_k\int_{\Delta_2} x^M q_k q_{k+l}\,d\beta\right| \\
&\quad\quad + \frac{1}{n}\sum_{k=0}^{n-1}\left|\frac{\gamma_{k-1}}{\gamma_k}\int_{\Delta_1} x^M p_{k-1} p_{k+l}\,d\alpha - \frac{\delta_{k-1}}{\delta_k}\int_{\Delta_2} x^M q_{k-1} q_{k+l}\,d\beta\right|. \quad (3.1)
\end{aligned}$$

Consider the middle term here, for example, and, for brevity, let us write the two integrals involved as $A_k(M:l)$ and $B_k(M:l)$ Then this term does not exceed

$$\frac{1}{n}\sum_{k=0}^{n-1} |\alpha_k| \, |A_k(M:l)-B_k(M:l)| + \frac{1}{n}\sum_{k=0}^{n-1} |B_k(M:l)| \, |\alpha_k-\beta_k|. \tag{3.2}$$

Since $\operatorname{supp}(d\alpha)$ is compact then $|\alpha_k| \leqslant K$ $(k=0, 1, 2, ...)$ and by hypothesis

$$\frac{1}{n}\sum_{k=0}^{n-1} |A_k(M:l)-B_k(M:l)| \to 0$$

as $n\to\infty$ for each integer $l\geqslant 0$. So the former sum in (3.2) is $o(1)$. Next

$$|B_k(M:l)| = \left|\int_{\Delta_2} x^M q_k q_{k+l}\, d\beta\right| \leqslant K_2(M)$$

$(l=0, 1, 2, ...)$ because of the compactness of Δ_2 and the Schwarz inequality. Then since

$$\frac{1}{n}\sum_{k=0}^{n-1} |\alpha_k-\beta_k| \to 0$$

the second term in (3.2) is also $o(1)$. The first and last terms in (3.1) are treated in exactly the same way and this completes the proof of Lemma 1.

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